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Numerical Approximation of the Maxwell Equations in Inhomogeneous Media by a P^1 Conforming Finite Element Method

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The aim of this paper is to present an extension of the P^1 conforming finite element method for the time-dependent Maxwell equations that has been previously exposed. We shall consider inhomogeneous media with the piecewise constant dielectric and magnetic parameters, and precisely, the interface of two such media. By analogy with the idea developed, we propose a method based on the dualization and then the approximation of the interface conditions in a way consistent with the one derived for the fields and the Lagrange multipliers of the divergence constraints inside the domain.

1. INTRODUCTION

In a previous paper [1], a P^1-conforming finite element method for the time-dependent three-dimensional Maxwell equations was proposed. The applications we had in mind involved electron beam propagation, hyperfrequency vacuum devices, etc. In this context, the method was required to deal with arbitrary geometries and was developed in view of coupling it with a particle solver of the Vlasov equation. However, the method proved to be also extremely worthwhile in pure electromagnetic wave propagation problems like time dependent radar problems [3]. Therefore, an extension of this method to inhomogeneous media is now investigated.

The first stage is to reexamine the choice of our methodology in the perspective of discontinuous fields at material interfaces. From now on, we restrict to the case of homogeneous media separated by abrupt interfaces. The treatment of media with a continuous variation of the dielectric and magnetic constants is somehow easier and can directly be adapted from [1]. Roughly speaking, the most widely spread methods for solving the Maxwell equations on unstructured meshes can be divided into two groups: the nodal and the edge element methods. Let us briefly review the advantages and drawbacks of each of them and explain the choice we make.

In the framework of the harmonic form of Maxwell's equations, one can refer to theoretical studies [4–6] as well as more numerical experiments [7–9] that compare various methods of discretization in 3D. In the literature, several problems like the appearance of spurious modes [10–12], the handling of inhomogeneous media and boundary conditions [5, 13, 14], or the ability to compute divergence-free fields [15–17] are investigated. For these questions, the advantages of the edge elements [5, 9, 18] were proved (see, among others, [19] for a review), even if this approach involves more unknowns than the nodal one [20–22]. The question of the accuracy for a given mesh is, up to our knowledge, not yet settled [21, 23, 24].

In the time dependent case, the situation is not so well established, and the last decade has seen an important literature about alternate methods, and comparisons with the standard ones [22, 25–28]. From their work, we can establish a list of criteria that should be satisfied by any numerical method for the Maxwell equations in the time domain:

1. "Good" dispersion properties are required (see [21] for a precise statement). Following [21], both edge and nodal elements can satisfy these requirements.

2. The handling of discontinuous dielectric and magnetic constants must be simple and easy. The edge element method (by construction [29, 30]) or the Lee–Madsen method [31] are well adapted, as they only impose the continuity of the tangential part of E and H, but not of the normal part [13, 32], as is physically required. Nevertheless, in regions of constant dielectric and magnetic parameters, the fields are smooth and the nodal elements, which use a continuous representation of the fields, should be better suited [33], because they are more accurate. For this reason, some authors [34] have investigated combinations of both edge and nodal elements, by devising some special adaptation of the finite element spaces.

3. Finally, the space and time discretizations should prevent the resolution of a linear system at each time-step. The edge element mass matrix is not diagonal and needs to be inverted or to be lumped by some specific technique. Lumping techniques have recently been developed in the framework of high order triangular nodal elements for the

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wave equation [35] and have been extended to the Maxwell equations with quadrilateral edge elements [36], but not yet with triangular elements (see, also in the same spirit, [37]). However, lumping techniques can lead to a reduction of accuracy. In this paper, we propose an alternative way, following the ideas developed in [1]: we use a nodal finite element method which naturally leads to a diagonal mass matrix. This paper specifically details the treatment of interface conditions between different media.

We first recall the method described in [1]. It is based on a wave equation formulation of the Maxwell equations coupled with the divergence constraints, and a P^1 conforming finite-element discretization based on the Hood–Taylor finite element [38]. A related method using Hood–Taylor-type finite elements can also be found in the framework of 3D eddy current problems in [39]. In this method, two tetrahedralizations are defined, the finer one being deduced from the coarser one by dividing each tetrahedron into eight subtetrahedra. The degrees of freedom of the method are the values of the fields at the nodes of the finer tetrahedralization and the values of the Lagrange multipliers of the divergence constraints at the nodes of the coarser one. Both the fields and the Lagrange multipliers are chosen to be continuous over the domain.

Now, if we consider two different homogeneous material media meeting along an interface, we require the tetrahedralization to follow the interface and we specify approximate values of the fields at the interface nodes, where the exact fields are discontinuous. Therefore, the approximate fields must also be discontinuous. We describe how this discontinuity can be handled.

The method is based on the dualization and, then, the approximation of the interface condition. A somehow-close method can be found in [40] for the treatment of boundary conditions in finite element approximations of the Navier–Stokes equations. We choose these approximations in a way consistent with those of the fields and of the Lagrange multipliers of the divergence constraints inside the domain. Then, we show that, under some approximation, the Lagrange mulipliers of the tangential transmission condition can be eliminated and we propose a way of dealing with the discrete constraints which appear after the elimination. The same method can somehow be used for the treatment of the normal transmission condition, but we show that it is more consistent not to eliminate the corresponding Lagrange multipliers. This paper is restricted to the treatment of an interface between two different media only. The case of more than two media which meet along an edge or at one point requires a special treatment which is presented in [41] and will be submitted in a forthcoming paper [42]. A detailed version of the algorithms developed in this paper can also be found in [41].

Section 2 is devoted to a rough outline of the method. The adequate variational formulation is proposed in Section 3 while Section 4 is devoted to the choice of the discrete spaces and to the convenient setting for the discrete variational formulation. Then, numerical examples are given in Section 5. A conclusion is drawn in Section 6. A detailed bibliography on the various approaches for solving the Vlasov–Maxwell equations in the time domain can be found in [1].

2. OUTLINE OF THE METHOD

In this paper, we shall consider a bounded domain Ω which consists of two different homogeneous media Ω_1 and Ω_2 separated by an interface Σ. We denote by $n(x)$, $x \in \Sigma$, the unit normal to Σ oriented from Ω_1 to Ω_2. Let $\varepsilon(x)$ and $\mu(x)$, $x \in \Omega$ be the dielectric permittivity and magnetic permeability, given by

$$\varepsilon(x) = \begin{cases} \varepsilon_1 & \text{in } \Omega_1 \\ \varepsilon_2 & \text{in } \Omega_2, \end{cases} \qquad \mu(x) = \begin{cases} \mu_1 & \text{in } \Omega_1 \\ \mu_2 & \text{in } \Omega_2, \end{cases}$$

where ε_i, μ_i, $i = 1, 2$, are constant scalars (the extension to constant tensors would be completely straightforward). The Maxwell equations are written

$$\frac{\partial}{\partial t}(\varepsilon E) - \nabla \times H = -J, \quad \frac{\partial}{\partial t}(\mu H) + \nabla \times E = 0, \quad x \in \Omega, \tag{2.1}$$

$$\nabla \cdot (\varepsilon E) = \rho, \quad \nabla \cdot (\mu H) = 0, \quad x \in \Omega. \tag{2.2}$$

The initial and boundary conditions will be specified later on. The existence of classical solutions to these equations is guaranteed as soon as the following jump relations hold across the surface of discontinuity of ε and μ (see, e.g., [43]),

$$\begin{aligned} [E \times n]_\Sigma = 0, \quad [H \times n]_\Sigma = 0, \\ [(\varepsilon E) \cdot n]_\Sigma = 0, \quad [(\mu H) \cdot n]_\Sigma = 0, \end{aligned} \tag{2.3}$$

where $[A]_\Sigma(x)$ denotes the jump of a quantity A across Σ at the point $x \in \Sigma$, namely,

$$[A]_\Sigma(x) = \lim_{y \to x, y \in \Omega_2} A(y) - \lim_{y \to x, y \in \Omega_1} A(y).$$

The solutions of the Cauchy problem for (2.1) satisfy the divergence constraints (2.2) as soon as the initial data satisfy these divergence constraints, on the one hand, and the continuity equation $\partial\rho/\partial t + \nabla \cdot J = 0$ is fulfilled, on the other hand. However, after discretization, these properties may very well be violated, for example, when the discrete charge and current density ρ_h and J_h are computed from

a particle approximation (see [44]), so that the corresponding discrete problem (2.1)–(2.2) becomes ill-posed. We shall follow the ideas of [1] and introduce a formulation of the Maxwell equations with dualization of the divergence constraints which guarantees the well-posedness of the problem. The formulation reads:

$$\begin{aligned} &\frac{\partial}{\partial t}(\varepsilon E) - \varepsilon\nabla\varphi - \nabla\times H = -J, \\ &\frac{\partial}{\partial t}(\mu H) - \mu\nabla p + \nabla\times E = 0, \quad x\in\Omega, \end{aligned} \tag{2.4}$$

together with (2.2), where $\varphi(x)$ and $p(x)$, for $x\in\Omega$ are the Lagrange multipliers of the divergence constraints (2.2). We shall specify $\varphi(x)$ and $p(x)$ to be in $H^1(\mathbb{R}^3)$, such that $\varepsilon\nabla\varphi$ and $\mu\nabla p$ belong to $H(\mathrm{div},\Omega)$ (see Section (3.3.1) for these definitions). Indeed, it is readily seen that the problem (2.4)–(2.2) is well-posed in these spaces, as soon as ρ and J belong to $L^2(\Omega)$. Note in particular that this well-posedness is not subject to the fulfillment by ρ and J of the continuity equation, because of the introduction of the multipliers. The regularity conditions on φ and p imply the following interface conditions:

$$[\varphi]_\Sigma = 0, \quad [p]_\Sigma = 0, \quad \left[\varepsilon\frac{\partial\varphi}{\partial n}\right]_\Sigma = 0, \quad \left[\mu\frac{\partial p}{\partial n}\right]_\Sigma = 0. \tag{2.5}$$

Notice that these interface conditions, together with Eqs. (2.4) imply the extra interface conditions,

$$[\varepsilon^{-1}(\nabla\times H - J)\times n]_\Sigma = 0, \quad [\mu^{-1}(\nabla\times E)\times n]_\Sigma = 0. \tag{2.6}$$

Now we outline the extension of the P^1 conforming finite element method described in [1] to material media. Let Σ_h be the surface which consists of faces of tetrahedra of the coarser mesh the vertices of which belong to Σ. We denote by $\mathcal{N}_C^\Sigma$ (respectively $\mathcal{N}_F^\Sigma$) the set of nodes of the coarser mesh (respectively finer mesh) which belong to Σ_h. We have $\mathcal{N}_C^\Sigma \subset \mathcal{N}_F^\Sigma$.

To account for the transmission conditions across Σ_h, the values of the fields E_h and H_h are doubled at the nodes of Σ_h, each value being assigned to one of the subdomains Ω_1 and Ω_2 (see Fig. 1). The approximate fields E_h and H_h are chosen continuous over each subdomain Ω_1 and Ω_2, with their restrictions to each tetrahedron of the finer mesh being polynomial of total degree 1. These approximate fields may still have different limits on Σ_h, depending on the subdomain Ω_1 and Ω_2 because of the doubling of the values of the fields at the nodes of Σ_h. These limits have to be connected by some approximation of the interface conditions (2.3). The aim of this paper is to rigorously derive the correct interface condition. For this purpose, a variational formalism will be set up.

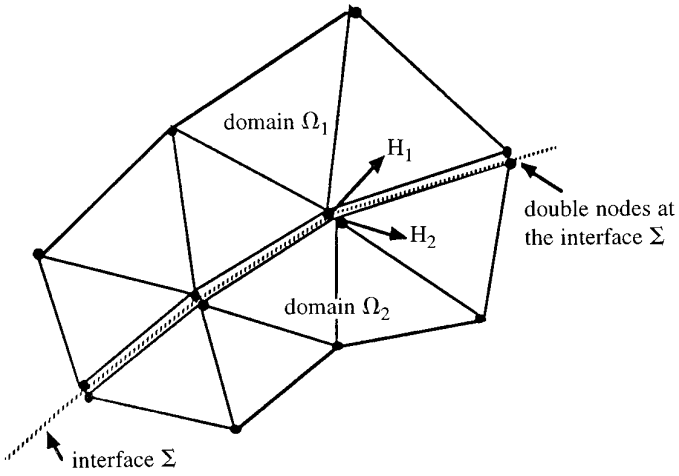

FIG. 1. Doubling the nodes at the interface Σ.

3. VARIATIONAL FORMULATIONS FOR THE CONSTRAINED MAXWELL SYSTEM

3.1. General Setting of the Problem

We are concerned by the Maxwell system (2.4), (2.2) in the domain Ω with initial conditions given by

$$E(x,0) = E_0(x), \quad H(x,0) = H_0(x), \quad x\in\Omega. \tag{3.1}$$

We assume that the boundary $\partial\Omega$ consists of a perfectly conducting boundary Γ_C and an artificial boundary Γ_A on which we prescribe the Silver–Müller absorbing boundary condition. Extension of the present work to higher order absorbing boundary conditions on Γ_A is possible but will not be considered here. Therefore, we prescribe

$$E\times n = 0, \quad x\in\Gamma_C, \tag{3.2}$$

$$(E - \sqrt{(\mu/\varepsilon)}\,H)\times n = (e - \sqrt{(\mu/\varepsilon)}\,h)\times n, \quad x\in\Gamma_A, \tag{3.3}$$

or, equivalently,

$$(H + \sqrt{(\varepsilon/\mu)}\,E)\times n = (h + \sqrt{(\varepsilon/\mu)}\,e)\times n, \quad x\in\Gamma_A, \tag{3.4}$$

where e and h are prescribed electric and magnetic fields along the artificial boundary Γ_A. The forms (3.3) and (3.4) are equivalent forms of the same boundary condition. We refer to [1] for more details about this boundary condition.

The Lagrange multipliers φ and p have to verify the following boundary conditions (which will be justified later on):

$$\varphi = 0, \quad x \in \partial\Omega; \quad \frac{\partial p}{\partial n} = 0, \quad x \in \Gamma_C; \quad p = 0, \quad x \in \Gamma_A. \tag{3.5}$$

3.2. The Wave Equation Formulation

Following [1], the first step is to derive a wave equation formulation of the Maxwell equations (2.4), (2.2). Elementary algebraic manipulations lead to

$$\frac{\partial^2(\mu H)}{\partial t^2} + \nabla \times (\varepsilon^{-1}\nabla \times H) - \mu\nabla P = \nabla \times (\varepsilon^{-1} J), \quad x \in \Omega, \tag{3.6}$$

$$\frac{\partial^2(\varepsilon E)}{\partial t^2} + \nabla \times (\mu^{-1}\nabla \times E) - \varepsilon\nabla\Phi = -\frac{\partial J}{\partial t}, \quad x \in \Omega, \tag{3.7}$$

together with (2.2) and

$$P = \frac{\partial p}{\partial t}, \quad \Phi = \frac{\partial \varphi}{\partial t}.$$

The initial data are given by

$$\begin{aligned} &H(x,0) = H_0(x), \\ &\frac{\partial H}{\partial t}(x,0) = \nabla p_0(x) + \mu^{-1}(\nabla \times E_0)(x), \quad x \in \Omega, \\ &E(x,0) = E_0(x), \\ &\frac{\partial E}{\partial t}(x,0) = \nabla \varphi_0(x) + \varepsilon^{-1}(\nabla \times H_0 - J_0)(x), \quad x \in \Omega \end{aligned} \tag{3.8, 3.9}$$

(where $\varphi_0(x) = \varphi(x,0)$ and $p_0(x) = p(x,0)$) and the boundary conditions are written

$$\varepsilon^{-1}(\nabla \times H - J) \times n = 0, \quad x \in \Gamma_C, \tag{3.10}$$

$$\left(\varepsilon^{-1}(\nabla \times H - J) - \sqrt{\frac{\mu}{\varepsilon}}\frac{\partial H}{\partial t}\right) \times n = \left(\frac{\partial e}{\partial t} - \sqrt{\frac{\mu}{\varepsilon}}\frac{\partial h}{\partial t}\right) \times n, \quad x \in \Gamma_A, \tag{3.11}$$

or

$$-\left(\mu^{-1}(\nabla \times E) + \sqrt{\frac{\varepsilon}{\mu}}\frac{\partial E}{\partial t}\right) \times n = \left(\frac{\partial h}{\partial t} + \sqrt{\frac{\varepsilon}{\mu}}\frac{\partial e}{\partial t}\right) \times n, \quad x \in \Gamma_A, \tag{3.12}$$

together with (3.5) and (3.2).

The combination of (3.6) with the boundary conditions (3.5) and (3.10) yields an extra boundary condition on H, namely,

$$\frac{\partial^2}{\partial t^2}(H \cdot n) = 0, \quad x \in \Gamma_C,$$

or using (3.8) and (3.5), (3.2),

$$H(x,t) \cdot n(x) = H_0(x) \cdot n(x) \quad \forall x \in \Gamma_C, \forall t > 0. \tag{3.13}$$

This extra boundary condition is of course satisfied by the solution of the initial Maxwell equations (2.1)–(2.2). Notice that it is satisfied by the constrained formulations (2.4), (2.2), or (3.6), (3.7), (2.2), only because we have chosen a Neumann boundary condition for P on Γ_C. In [1], a Dirichlet boundary condition for P was chosen on Γ_C, so that (3.13) was not satisfied, which yielded a less accurate solution near the boundary for the magnetic field.

3.3. Variational Formulation with Dualization of the Transmission Constraints

3.3.1. *Notations*

As usual, we have the following canonical inclusions [45]:

$$H^{1/2}(\Gamma_C) \subset L^2(\Gamma_C) \subset H^{-1/2}(\Gamma_C).$$

For a distribution $T \in H^{-1/2}(\Gamma_C)$ and a function $f \in H^{1/2}(\Gamma_C)$, we denote by $(T, f)_{\Gamma_C}$, the duality pairing. The same remarks and notations are also relevant for Γ_A and for the interface Σ. The symbol $(B, C)_\Omega$ stands for the $L^2(\Omega)$ inner product of the two vector fields B and C, namely

$$(B, C)_\Omega = \int_\Omega B(x) \cdot C(x)\, dx,$$

where $\cdot$ stands for the usual inner product of vectors in $\mathbb{R}^3$. We also recall [45]

$$\begin{aligned} H(\mathrm{curl}, \Omega) &= \{B \in L^2(\Omega)^3, \nabla \times B \in L^2(\Omega)^3\}, \\ H(\mathrm{div}, \Omega) &= \{B \in L^2(\Omega)^3, \nabla \cdot B \in L^2(\Omega)\}, \end{aligned}$$

and $H(\mathrm{curl}, \mathrm{div}, \Omega) = H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega)$. We recall that the scalar components of the tangential traces of fields in $H(\mathrm{curl}, \Omega)$ along the boundary, as well as the normal traces of fields of $H(\mathrm{div}, \Omega)$ are defined in $H^{-1/2}(\partial\Omega)$. The trace property is also true for interfaces inside the domain and gives a meaning to the interface conditions. We refer to [45] for a more extensive discussion of the trace properties of the spaces $H(\mathrm{curl}, \Omega)$ and $H(\mathrm{div}, \Omega)$. We define

$$\overline{H}(\Omega) = \{H : \Omega \to \mathbb{R}^3, H|_{\Omega_i} \in H(\mathrm{curl}, \mathrm{div}, \Omega_i), i = 1, 2\}. \tag{3.14}$$

Then, we let $H^{1/2}(N\Sigma)$ and $H^{1/2}(T\Sigma)$ be respectively the spaces of normal and tangent vector fields to Σ with

regularity $H^{1/2}$. These spaces will be respectively the functional spaces for the Lagrange multipliers of the normal and tangential transmission conditions. We also introduce

$$\Gamma_i = \partial\Omega \cap \overline{\Omega_i}, \quad \Gamma_{Ai} = \Gamma_A \cap \overline{\Omega_i}, \quad \Gamma_{Ci} = \Gamma_C \cap \overline{\Omega_i}, \quad i = 1, 2,$$

and we denote by n_{Ai} and n_{Ci} the outward unit normals to Γ_{Ai} and Γ_{Ci}. We shall denote with an index C (resp. A) the various intersections of the medium Ω_i with the perfectly conducting boundary Γ_C (resp. Γ_A). Then, the spaces of the Lagrange multipliers of the normal and tangential boundary conditions on Γ_C will be chosen to be $\Pi_{i=1}^2 H^{1/2}(N\Gamma_{Ci})$ and $\Pi_{i=1}^2 H^{1/2}(T\Gamma_{ci})$, respectively.

3.3.2. *The Variational Formulation with Dualization of the Transmission Constraints for the H Field*

This formulation reads:

"Find $H \in \overline{H}(\Omega)$, $P \in L^2(\Omega)$, $\nu \in H^{1/2}(N\Sigma)$, $\tau \in H^{1/2}(T\Sigma)$, $\nu_C = (\nu_{C1}, \nu_{C2}) \in \Pi_{i=1}^2 H^{1/2}(N\Gamma_{Ci})$:

$$\begin{aligned}
&\frac{d^2}{dt^2}\sum_{i=1}^{2}\mu_i(H,K)_{\Omega_i} + \frac{d}{dt}\sum_{i=1}^{2}\sqrt{\frac{\mu_i}{\varepsilon_i}}(H_i\times n_{Ai}, K_i\times n_{Ai})_{\Gamma_{Ai}} \\
&\quad + \sum_{i=1}^{2}\varepsilon_i^{-1}(\nabla\times H, \nabla\times K)_{\Omega_i} \\
&\quad + \sum_{i=1}^{2}\mu_i(\nabla\cdot K, P)_{\Omega_i} + \sum_{i=1}^{2}\mu_i(K_i, \nu_{Ci})_{\Gamma_{Ci}} \\
&\quad + (\nu, \mu_1 K_1 - \mu_2 K_2)_\Sigma \\
&\quad + (\tau, (K_1 - K_2)\times n)_\Sigma = \sum_{i=1}^{2}\varepsilon_i^{-1}(J, \nabla\times K)_{\Omega_i} \\
&\quad - \frac{d}{dt}\sum_{i=1}^{2}\left(\left(e_i - \sqrt{\frac{\mu_i}{\varepsilon_i}}h_i\times n_{Ai}\right), K_i\times n_{Ai}\right)_{\Gamma_{Ai}} \\
&\qquad\qquad \forall K \in \overline{H}(\Omega), \qquad (3.15)
\end{aligned}$$

$$(\nabla\cdot H, q_i)_{\Omega_i} = 0 \quad \forall q_i \in L^2(\Omega_i), \forall i = 1, 2, \qquad (3.16)$$

$$(H_i, \lambda_{Ci})_{\Gamma_{Ci}} = (H_{0i}, \lambda_{Ci})_{\Gamma_{Ci}} \quad \forall \lambda_{Ci} \in H^{1/2}(N\Gamma_{Ci}), \forall i = 1, 2, \qquad (3.17)$$

$$(\lambda, \mu_1 H_1 - \mu_2 H_2)_\Sigma = 0 \quad \forall \lambda \in H^{1/2}(N\Sigma), \qquad (3.18)$$

$$(\sigma, (H_1 - H_2)\times n)_\Sigma = 0 \quad \forall \sigma \in H^{1/2}(T\Sigma)." \qquad (3.19)$$

Note that ν is a normal vector field and that, for instance, $(K_i, \nu_{Ci})_{\Gamma_{Ci}} = (K_i\cdot n_{Ci}, \nu_{Ci}\cdot n_{Ci})$. It is easily checked that problem (3.15)–(3.19) is a variational formulation of the wave equations (3.6), (3.7), (2.2) with boundary conditions (3.10), (3.11), (3.5) (a detailed proof can be found in [41]). The interpretation of the Lagrange multipliers ν and τ will be useful for the numerical approximation. By using the Green formula on (3.15) with a test function K in $\overline{H}(\Omega)$ which vanishes on the boundary and on the subdomain Ω_2 and which is such that $K_1\cdot n = 0$ and $K_1\times n$ is arbitrary on Σ, we obtain

$$\varepsilon_1^{-1}(\nabla\times H_1, n\times K_1)_\Sigma + (\tau, K_1\times n)_\Sigma = \varepsilon_1^{-1}(J, n\times K_1)_\Sigma,$$

which leads to

$$\tau\times n = (\varepsilon_1^{-1}((\nabla\times H_1) - J_1)\times n)_\Sigma.$$

By interchanging 1 and 2, we also have

$$\tau\times n = (\varepsilon_2^{-1}((\nabla\times H_2) - J_2)\times n)_\Sigma.$$

This implies the interface condition

$$[(\varepsilon^{-1}((\nabla\times H) - J)\times n)]_\Sigma = 0, \qquad (3.20)$$

which is exactly the first condition (2.6). Furthermore, $\tau\times n$ appears to be the common value of the traces of $\varepsilon^{-1}((\nabla\times H) - J)\times n$ on both sides of Σ. These traces are also equal to the common value of the traces of $(\partial E/\partial t)\times n$ on both sides of Σ. Therefore, we have

$$\tau\times n = \left(\frac{\partial E_1}{\partial t}\times n\right)\Big|_\Sigma = \left(\frac{\partial E_2}{\partial t}\times n\right)\Big|_\Sigma. \qquad (3.21)$$

Conditions (2.6) or (3.21) express that there is no (time derivative of) magnetic current on Σ. Of course, all these calculations are only justified if the fields are regular enough. In general, the regularity given by the variational formulation alone is not sufficient to allow the integrations by parts which lead to (3.21). Therefore we would rather view it as an "interpretation" of the Lagrange multiplier $\tau\times n$ rather than an exact formula. This interpretation is fundamental for the treatment of intersections of more than two media along edges or at points. Indeed, in these cases, formulas (3.21) will be used to link together the several values of the Lagrange multipliers (as many as the number of interfaces meeting along this edge or at this point). Indeed, in [41], we detail why these Lagrange multipliers cannot be left independent one to each other.

In a very analogous fashion, we have

$$\nu\cdot n = -P_1|_\Sigma = -P_2|_\Sigma, \qquad (3.22)$$

so that the interface relation (2.5) holds, or in other words, there is no magnetic charge at the interface Σ. Furthermore, (3.22) shows that $\nu\cdot n$ can be interpreted as the common value of the trace of P on both sides of the interface Σ. Similarly, we have

$$\nu_{Ci} \cdot n_{Ci} = -P_i|_{\Gamma_{Ci}}, \quad i = 1, 2, \tag{3.23}$$

which shows that $\nu_{Ci} \cdot n_{Ci}$ can be interpreted as the trace of P on Γ_{Ci}.

3.3.3. *The Variational Formulation with Dualization of the Transmission Constraints for the E Field*

The formulation reads:

"Find $E \in \overline{H}(\Omega)$, $\Phi \in L^2(\Omega)$, $\nu \in H^{1/2}(N\Sigma)$, $\tau \in H^{1/2}(T\Sigma)$, $\tau_C = (\tau_{C1}, \tau_{C2}) \in \Pi_{i=1}^2 H^{1/2}(T\Gamma_{Ci})$:

$$\begin{aligned}
&\frac{d^2}{dt^2}\sum_{i=1}^{2} \varepsilon_i(E, F)_{\Omega_i} + \frac{d}{dt}\sum_{i=1}^{2} \sqrt{\frac{\varepsilon_i}{\mu_i}}(E_i \times n_{Ai}, F_i \times n_{Ai})_{\Gamma_{Ai}} \\
&\quad + \sum_{i=1}^{2} \mu_i^{-1}(\nabla \times E, \nabla \times F)_{\Omega_i} \\
&\quad + \sum_{i=1}^{2} \varepsilon_i(\nabla \cdot F, \Phi)_{\Omega_i} + \sum_{i=1}^{2} (F_i \times n_{Ci}, \tau_{Ci})_{\Gamma_{Ci}} \\
&\quad + (\nu, \varepsilon_1 F_1 - \varepsilon_2 F_2)_\Sigma \\
&\quad + (\tau, (F_1 - F_2) \times n)_\Sigma = -\frac{d}{dt}\sum_{i=1}^{2} (J, F)_{\Omega_i} \\
&\quad + \frac{d}{dt}\sum_{i=1}^{2} \left(\left(h_i + \sqrt{\frac{\varepsilon_i}{\mu_i}} e_i \times n_{Ai} \right), F_i \times n_{Ai} \right)_{\Gamma_{Ai}} \\
&\qquad\qquad \forall F \in \overline{H}(\Omega),
\end{aligned} \tag{3.24}$$

$$(\nabla \cdot E, \psi_i)_{\Omega_i} = 0 \quad \forall \psi_i \in L^2(\Omega_i), i = 1, 2, \tag{3.25}$$

$$(E_i \times n_i, \sigma_{Ci})_{\Gamma_{Ci}} = 0 \quad \forall \sigma_{Ci} \in H^{1/2}(T\Gamma_{Ci}), i = 1, 2, \tag{3.26}$$

$$(\lambda, \varepsilon_1 E_1 - \varepsilon_2 E_2)_\Sigma = 0 \quad \forall \lambda \in H^{1/2}(N\Sigma), \tag{3.27}$$

$$(\sigma, (E_1 - E_2) \times n)_\Sigma = 0 \quad \forall \sigma \in H^{1/2}(T\Sigma)." \tag{3.28}$$

Remark 3.1. The Lagrange multipliers ν and τ in the E field and H field formulations are not the same.

The interpretation of the Lagrange multipliers follows from the same method as for H. This gives the formal identities

$$\begin{aligned}
\tau \times n &= (\mu_1^{-1}(\nabla \times E_1) \times n)|_\Sigma \\
&= (\mu_2^{-1}(\nabla \times E_2) \times n)|_\Sigma
\end{aligned} \tag{3.29}$$

and, thus,

$$[\mu^{-1}(\nabla \times E) \times n]_\Sigma = 0, \tag{3.30}$$

which exactly yields the second interface condition (2.6). Furthermore, (3.29) can be rewritten

$$\tau \times n = \left(\frac{\partial H_1}{\partial t} \times n \right)\Big|_\Sigma = \left(\frac{\partial H_2}{\partial t} \times n \right)\Big|_\Sigma \tag{3.31}$$

and shows that $\tau \times n$ coincides (at least formally) with the common value of the (time derivative of) $H \times n$ on both sides of the interface. Relation (3.31) can be interpreted as the absence of electric current along the interface Σ.

Then, we have

$$\nu \cdot n = -\Phi_1|_\Sigma = -\Phi_2|_\Sigma, \tag{3.32}$$

so that the interface relation (2.5) holds, or in other words, there is no electric charge at the interface Σ. $\nu \cdot n$ can be interpreted as the common value of the trace of Φ on both sides of the interface Σ. We also have

$$\begin{aligned}
\tau_{Ci} \times n_{Ci} &= (\mu_i^{-1}(\nabla \times E_i) \times n_{Ci})|_{\Gamma_{Ci}} \\
&= \left(\frac{\partial H_i}{\partial t} \times n_{Ci} \right)\Big|_{\Gamma_{Ci}},
\end{aligned} \tag{3.33}$$

which shows that τ_{Ci} can be interpreted as the trace of $(\partial H/\partial t) \times n$, or equivalently, as the (time derivative of the) electric current on Γ_{Ci}. From these considerations, showing that the solution of formulation (3.24)–(3.28) satisfies the correct interface and boundary conditions is very easy.

3.3.4. *Replacement of the Curl–Curl Operator by a Laplace Operator*

If the fields are regular enough, we can replace the space $\overline{H}(\Omega)$ by the space $H(\Omega)$ defined by

$$H(\Omega) = \{H \in L^2(\Omega)^3, H|_{\Omega_i} \in H^1(\Omega_i)\}. \tag{3.34}$$

Then, for H and K in $H(\Omega)$, we have [46] (with the Einstein summation conventions)

$$\begin{aligned}
&(\nabla \times H, \nabla \times K)_{\Omega_i} + (\nabla \cdot H, \nabla \cdot K)_{\Omega_i} \\
&\quad = (\nabla : H, \nabla : K)_{\Omega_i} + (\nabla H_{i,\alpha} \times n_{ij}, u_\alpha \times K_i)_\Sigma \\
&\quad + \sum_{i=1}^{2} (\nabla H_{i,\alpha} \times n_i, u_\alpha \times K_i)_{\Gamma_i},
\end{aligned} \tag{3.35}$$

where the double dots : denote the contracted product of two tensors, $n_{ij} = n$ if $(i, j) = (1, 2)$ and $-n$ if $(i, j) = (2,$

1), $H_{i,\alpha}$, is the αth component of the field H_i and u_α is the basis vector in the αth direction. However, it must be pointed out that the formulations with $H(\Omega)$ and $\overline{H}(\Omega)$ are not equivalent in general when the fields are not regular enough, i.e., when the geometry is singular. The solutions of the formulation posed on $H(\Omega)$ are not singular although the true solutions of the Maxwell equations are. However, the singularity moves to the Lagrange multipliers of the divergence constraints (P, Φ, and ν_C) which can no more be proved to be zero. In this respect, the modified formulation in the space $H(\Omega)$ amounts to project the solution of Maxwell's equation on the regular fields and to carry the singularity onto the Lagrange multipliers. In practice, we use the formulation in $H(\Omega)$.

4. THE DISCRETE VARIATIONAL FORMULATION

4.1. The Mesh

For the sake of simplicity, the perfectly conducting boundary Γ_C and the external applied fields e and h will be ignored in the forthcoming presentation. Their treatment is rather easy and we refer the reader to [41 or 42] for more details. As noted in Section 2, we introduce two levels of meshes. A coarser tetrahedral mesh and a finer one, obtained by dividing each tetrahedron into eight subtetrahedra. We assume that the tetrahedralization follows the interface Σ, in the sense that Σ can be approximated by a surface Σ_h which consists of a union of faces of tetrahedra of either mesh.

We now introduce the following notations. $\mathscr{T}_C^{\Omega}$ denotes the set of the coarser tetrahedra of $\overline{\Omega}$ while $\mathscr{T}_F^{\Omega}$ denotes the set of the finer ones. The similar notations $\mathscr{T}_C^{\Omega_i}$ and $\mathscr{T}_F^{\Omega_i}$ hold for the subset of tetrahedra which are contained in $\overline{\Omega_i}$. By the above assumption we have

$$\mathscr{T}_C^{\Omega_1} \cup \mathscr{T}_C^{\Omega_2} = \mathscr{T}_C^{\Omega}, \quad \mathscr{T}_F^{\Omega_1} \cup \mathscr{T}_F^{\Omega_2} = \mathscr{T}_F^{\Omega}.$$

We also denote $\mathscr{N}_C^{\Omega}$ and $\mathscr{N}_F^{\Omega}$ the set of nodes of the coarser mesh and of the finer mesh, respectively, in $\overline{\Omega}$, and $\mathscr{N}_C^{\Omega_i}$, $\mathscr{N}_F^{\Omega_i}$, the similar sets for each of the subdomains $\overline{\Omega_i}$. Also, $\mathscr{N}_C^{\Sigma}$ and $\mathscr{N}_F^{\Sigma}$ stand for the set of the nodes of the coarser and finer mesh, respectively, which belong to the approximation Σ_h of the surface Σ. By definition, we have

$$\mathscr{N}_C^{\Omega_1} \cap \mathscr{N}_C^{\Omega_2} = \mathscr{N}_C^{\Sigma}, \quad \mathscr{N}_F^{\Omega_1} \cap \mathscr{N}_F^{\Omega_2} = \mathscr{N}_F^{\Sigma}.$$

Finally, we denote by $\mathscr{T}_C^{\Sigma}$ and $\mathscr{T}_F^{\Sigma}$ the traces respectively of the coarser and finer tetrahedralization of $\overline{\Omega}$ on Σ, namely the set of triangles which are faces of tetrahedra of the coarser or finer mesh and which are contained in Σ_h. From now on, we identify the surface Σ and its approximation Σ_h.

Let y be a node of the finer tetrahedralization of $\overline{\Omega_i}$ (i.e., $y \in \mathscr{N}_F^{\Omega_i}$). We denote by ϕ_y^i the P^1 basis function associated with y with respect to the finer mesh restricted to $\overline{\Omega_i}$. If, $y \in \Sigma$, there exist two such basis functions associated with y: ϕ_y^1 and ϕ_y^2, which have disjoint supports. Similarly, if z is a node of the coarser tetrahedrization of $\overline{\Omega_i}$, we define ψ_z^i, the P^1 basis function associated with z with respect to the coarser mesh restricted to $\overline{\Omega_i}$. Throughout the paper, y will denote nodes of the finer tetrahedrization while z will stand for nodes of the coarser one.

4.2. The Discrete Spaces

We define $V^h \subset H(\Omega)$ by

$$\begin{aligned} V^h = \{&H \in H(\Omega),\ H_i = H|_{\overline{\Omega_i}} \in \mathscr{C}^0(\overline{\Omega_i})^3,\ i = 1, 2,\\ &H_{i|T} \in P^1(T),\ \forall T \in \mathscr{T}_F^{\Omega_i},\ i = 1, 2\}. \end{aligned} \tag{4.1}$$

For $H \in V^h$, we can write

$$H(x) = \sum_{y \in \mathscr{N}_F^{\Omega_1}} H_y^1 \phi_y^1(x) + \sum_{y \in \mathscr{N}_F^{\Omega_2}} H_y^2 \phi_y^2(x), \tag{4.2}$$

where $H_y^i \in \mathbb{R}^3$, for any node y of the finer tetrahedralization of $\overline{\Omega_i}$. We also define $L^{2h} \subset L^2(\Omega)$ by

$$\begin{aligned} L^{2h} = \{&P \in L^2(\Omega),\ P_i = P|_{\overline{\Omega_i}} \in \mathscr{C}^0(\overline{\Omega_i}),\ i = 1, 2,\\ &P_{i|T} \in P^1(T),\ \forall T \in \mathscr{T}_C^{\Omega_i},\ i = 1, 2\}. \end{aligned} \tag{4.3}$$

For $P \in L^{2h}$, we can write

$$P(x) = \sum_{z \in \mathscr{N}_C^{\Omega_1}} P_z^1 \psi_z^1(x) + \sum_{z \in \mathscr{N}_C^{\Omega_2}} P_z^2 \psi_z^2(x), \tag{4.4}$$

where $P_z^i \in \mathbb{R}$, for any node z of the coarser tetrahedralization of $\overline{\Omega_i}$.

For the Lagrange multipliers of the tangential constraints on Σ, we choose a space of continuous functions in order to have the same regularity as for the Lagrange multipliers inside the domain. We recall that $H^{1/2}(T\Sigma)$ consists of vector fields tangent to Σ. However, it is not possible in general to find smooth approximate fields tangent to the surface Σ_h because Σ_h has angular points at the nodes $z \in \mathscr{N}_C^{\Sigma}$. We thus emphasize the smoothness condition and relax the geometrical condition by defining an external approximation $T^h \not\subset H^{1/2}(T\Sigma)$.

Let $y \in \mathscr{N}_F^{\Sigma}$. We define the normal vector of Σ_h at y by

$$n_y^{\Sigma} = \sum_{T \in \mathscr{T}_F^{\Sigma}(y)} n_T \operatorname{Surf}(T) \Big/ \sum_{T \in \mathscr{T}_F^{\Sigma}(y)} \operatorname{Surf}(T), \tag{4.5}$$

where $\mathscr{T}_F^\Sigma(y)$ denotes the subset of triangles of $\mathscr{T}_F^\Sigma$ which share y as a common vertex, and n_T is the unit normal to the triangle T oriented from Ω_1 to Ω_2. Note that n_y^Σ is not a unit vector. A similar formula is used to define n_z^Σ, for $z \in \mathscr{N}_C^\Sigma$, by simply replacing the set $\mathscr{T}_F^\Sigma(y)$, by its analogue for the coarser mesh $\mathscr{T}_C^\Sigma(z)$.

Then, for $y \in \mathscr{T}_F^\Sigma$, we define ϕ_y^Σ as the restriction of Σ of the P^1-basis function associated with the finer triangulation and the node y. We then define the approximation space T^h of $H^{1/2}(T\Sigma)$ in the following way:

$$T^h = \left\{ \tau \in \mathscr{C}^0(\Sigma)^3,\ \tau(x) = \sum_{y \in \mathscr{N}_F^\Sigma} \tau_y \phi_y^\Sigma(x),\ \tau_y \in \mathbb{R}^3,\ \tau_y \cdot n_y^\Sigma = 0,\ \forall y \in \mathscr{N}_F^\Sigma \right\}. \tag{4.6}$$

We notice that $T^h \not\subset H^{1/2}(T\Sigma)$ because $\tau(x) \cdot n(x) \neq 0$, in general, if $n(x)$ is the normal to Σ_h.

Similarly, for $z \in \mathscr{T}_C^\Sigma$, we define ψ_z^Σ as the restriction to Σ of the P^1-basis function associated with the coarser triangulation and the node z. We then define the approximation space N^{2h} of $H^{1/2}(N\Sigma)$ in the following way:

$$N^{2h} = \left\{ \nu \in \mathscr{C}^0(\Sigma)^3,\ \nu(x) = \sum_{z \in \mathscr{N}_C^\Sigma} \nu_z n_z^\Sigma \psi_z^\Sigma(x),\ \nu_z \in \mathbb{R},\ \forall z \in \mathscr{N}_C^\Sigma \right\}. \tag{4.7}$$

We notice that $N^{2h} \not\subset H^{1/2}(N\Sigma)$ because $\nu(x) \times n(x) \neq 0$, in general.

With these definitions, τ is chosen P^1-conforming on the trace on the finer tetrahedralization, while ν is chosen P^1-conforming on the coarser tetrahedralization. The reason for doing so is that τ is interpreted through relation (3.21) as the tangential trace on Σ of a field while ν is interpreted through (3.22) as the trace on Σ of a Lagrange multiplier.

Now, the numerical method is defined by the discrete variational formulations obtained by replacing in (3.15)–(3.19) and (3.24)–(3.28) the spaces $H(\Omega)$, $L^2(\Omega)$, $H^{1/2}(N\Sigma)$, and $H^{1/2}(T\Sigma)$ respectively by V^h, L^{2h}, N^{2h}, and T^h.

4.3. Time Discretization and Resolution

The time discretization is obtained by the usual leap-frog scheme. Let H^n, P^n, ν^n, τ^n, and J^n be the approximations of H, P, ν, τ, and J at time $t^n = n\,\Delta t$. Then the time discretization of formulation (3.15)–(3.19) (with, $\Gamma_C = \varnothing$, $e = h = 0$) is given by

"Find $H^{n+1} \in V^h$, $P^{n+1} \in L^{2h}$, $\nu^{n+1} \in N^{2h}$, $\tau^{n+1} \in T^h$:

$$\sum_{i=1}^2 \mu_i \left(\frac{H^{n+1} - 2H^n + H^{n-1}}{\Delta t^2}, K \right)_{\Omega_i} + \sum_{i=1}^2 \sqrt{\frac{\mu_i}{\varepsilon_i}} \left(\frac{H_i^{n+1} - H_i^{n-1}}{2\Delta t} \times n_i, K_i \times n_i \right)_{\Gamma_{Ai}} + \sum_{i=1}^2 \varepsilon_i^{-1} (\nabla \times H^n, \nabla \times K)_{\Omega_i} + \sum_{i=1}^2 \mu_i (\nabla \cdot K, P^{n+1})_{\Omega_i} + (\nu^{n+1}, \mu_1 K_1 - \mu_2 K_2)_\Sigma \times (\tau^{n+1}, (K_1 - K_2) \times n)_\Sigma = \sum_{i=1}^2 \varepsilon_i^{-1} (J^n, \nabla \times K)_{\Omega_i} \quad \forall K \in V^h, \tag{4.8}$$

$$\sum_{i=1}^2 \mu_i (\nabla \cdot H^{n+1}, q)_{\Omega_i} = 0 \quad \forall q \in L^{2h}, \tag{4.9}$$

$$(\lambda, \mu_1 H_1^{n+1} - \mu_2 H_2^{n+1})_\Sigma = 0 \quad \forall \lambda \in H^{2h}, \tag{4.10}$$

$$(\sigma, (H_1^{n+1} - H_2^{n+1}) \times n)_\Sigma = 0 \quad \forall \sigma \in T^h.\text{"} \tag{4.11}$$

Let $\tilde{P}^{n+1} = \Delta t^2 P^{n+1}$, $\tilde{\nu}^{n+1} = \Delta t^2 \nu^{n+1}$, $\tilde{\tau}^{n+1} = \Delta t^2 \tau^{n+1}$. Then, H^{n+1}, $\tilde{P}^{n+1}$, $\tilde{\nu}^{n+1}$, and $\tilde{\tau}^{n+1}$ solve the variational formulation:

"Find $H \in V^h$, $P \in L^{2h}$, $\nu \in N^{2h}$, $\tau \in T^h$:

$$\sum_{i=1}^2 \mu_i (H, K)_{\Omega_i} + \sum_{i=1}^2 \frac{\Delta t}{2} \sqrt{\frac{\mu_i}{\varepsilon_i}} (H_i \times n_i, K_i \times n_i)_{\Gamma_{Ai}} + \sum_{i=1}^2 \mu_i (\nabla \cdot K, P)_{\Omega_i} + (\nu, \mu_1 K_1 - \mu_2 K_2)_\Sigma + (\tau, (K_1 - K_2) \times n)_\Sigma = (G_H, K)_\Omega \quad \forall K \in V^h, \tag{4.12}$$

and such that Eqs. (4.9)–(4.11) are satisfied."

In (4.12), we dropped the tildes and the superscripts "$n + 1$" for simplicity, and $(G_h, K)_\Omega$ is a global notation for all the other terms of Eq. (4.8) which do not appear in the left-hand side of (4.12). Note that since V^h is a finite dimensional space, G_H is a linear form on V^h, which justifies the duality notation.

Then, the resolution algorithm is better set if Eqs. (4.12) and (4.9)–(4.11) are written in matrix form. For this purpose, we introduce a positive definite self-adjoint operator M (for "mass matrix") on V^h, defined by

$$(MH, K)_\Omega = \sum_{i=1}^2 \mu_i (H, K)_{\Omega_i} + \sum_{i=1}^2 \frac{\Delta t}{2} \sqrt{\frac{\mu_i}{\varepsilon_i}} (H_i \times n_i, K_i \times n_i)_{\Gamma_{Ai}} \quad \forall H, K \in V^h. \tag{4.13}$$

Similarly, we define the operators $L : L^{2h} \to V^h$, $N : N^{2h} \to V^h$, and $T : T^h \to V^h$ by

$$(LP, K)_\Omega = \sum_{i=1}^{2} \mu_i (\nabla \cdot K, P)_{\Omega_i} \quad \forall K \in V^h, \forall P \in L^{2h}, \tag{4.14}$$

$$(N\nu, K)_\Omega = (\nu, \mu_1 K_1 - \mu_2 K_2)_\Sigma \quad \forall K \in V^h, \forall \nu \in N^{2h}, \tag{4.15}$$

$$(T\tau, K)_\Omega = (\tau, (K_1 - K_2) \times n)_\Sigma \quad \forall K \in V^h, \forall \tau \in T^h. \tag{4.16}$$

These operators allow us to write system (4.12) and (4.9)–(4.11) in the form

$$\begin{aligned} MH + LP + N\nu + T\tau &= G_H \\ L^T H &= 0 \\ N^T H &= 0 \\ T^T H &= 0. \end{aligned} \tag{4.17}$$

Since M is a positive definite self-adjoint operator, we can write

$$H = M^{-1}(G_H - LP - N\nu - T\tau). \tag{4.18}$$

This allows us to eliminate H from system (4.17) and to end up with the following matrix system for P, ν, and τ:

$$\begin{aligned} (L^{\mathrm{T}}M^{-1}L)P + (L^{\mathrm{T}}M^{-1}N)\nu + (L^{\mathrm{T}}M^{-1}T)\tau &= L^{\mathrm{T}}M^{-1}G_H \\ (N^{\mathrm{T}}M^{-1}L)P + (N^{\mathrm{T}}M^{-1}N)\nu + (N^{\mathrm{T}}M^{-1}T)\tau &= N^{\mathrm{T}}M^{-1}G_H \\ (T^{\mathrm{T}}M^{-1}L)P + (T^{\mathrm{T}}M^{-1}N)\nu + (T^{\mathrm{T}}M^{-1}T)\tau &= T^{\mathrm{T}}M^{-1}G_H. \end{aligned} \tag{4.19}$$

If the variational formulation (4.12) and (4.9)–(4.11) is well-posed, the matrix system (4.19) is invertible. The well-posedness of the discrete variational formulation and the convergence of the method will be investigated in future work.

Remark 4.1. In the vacuum case, only the upper left block of the matrix system (4.19) appears in the matrix formulation. The system is solved by a conjugate gradient method with a diagonal preconditioner.

By analogy with the vacuum case, the numerical resolution of system (4.19) is more efficiently done by a block iterative method. The inversion of each block $(L^{\mathrm{T}}M^{-1}L)$, $(N^{\mathrm{T}}M^{-1}N)$, or $(T^{\mathrm{T}}M^{-1}T)$ is done by a conjugate gradient method with a diagonal preconditioner. We recall that we do not need to reach high levels of convergence, since the Lagrange multipliers are not physically interesting quantities. We only need to ensure that at each time step, the field will not go too far away from the subspace defined by the divergence and transmission constraints.

Even with the above described algorithm, the inversion of system (4.19) and the storage of the Lagrange multipliers are too expensive in terms of computer resources. However, there is an easy way of eliminating at least the Lagrange mulipliers τ of the tangential constraints, by use of an appropriate quadrature formula. The elimination of the Lagrange multipliers ν of the normal constraints may also be done to some extent. These methods are detailed in the next subsection.

4.4. Elimination of the Lagrange Multipliers of the Transmission Constraints

4.4.1. *An Approximation of the Bilinear Form* $(\tau, (H_1 - H_2) \times n)_\Sigma$

We begin to write

$$\begin{aligned} &(\tau, (H_1 - H_2) \times n)_\Sigma \\ &\quad = \int_\Sigma \left(\sum_{y \in \mathcal{N}_F^\Sigma} \tau_y \phi_y^\Sigma(x) \right) \\ &\qquad \cdot \left(\sum_{y \in \mathcal{N}_F^\Sigma} ((H_y^2 - H_y^1) \times n)\, \phi_y^\Sigma(x) \right) dS(x). \end{aligned}$$

If one decomposes the integral according to the triangulation of Σ, we get

$$\begin{aligned} (\tau, (H_1 - H_2) \times n)_\Sigma = \sum_{T \in \mathcal{T}_F^\Sigma} \sum_{y, y' \in \mathcal{N}_F^\Sigma(T)} \\ (\tau_y, (H_{y'}^2 - H_{y'}^1) \times n_T) \int_T \phi_y^\Sigma(x) \phi_{y'}^\Sigma(x)\, dS(x), \end{aligned}$$

where, for a triangle T in $\mathcal{T}_F^\Sigma$ we denote by $\mathcal{N}_F^\Sigma(T)$ the set of nodes of the finer tetrahedralization which are vertices of T. We recall the quadrature formula, exact for polynomials of degree one,

$$\int_T g(x)\, dS(x) \simeq \frac{1}{3} \left(\sum_{y \text{ vertex of } T} g(y) \right) \mathrm{Surf}(T). \tag{4.20}$$

By using this quadrature formula, we have

$$\int_T \phi_y^\Sigma(x)\, \phi_{y'}^\Sigma(x)\, dS(x) = \frac{1}{3} \mathrm{Surf}(T)\, \delta_{yy'},$$

where δ stands for the Kronecker delta. Therefore,

$$\begin{aligned} &(\tau, (H_1 - H_2) \times n)_\Sigma \\ &\quad \simeq \sum_{T \in \mathcal{T}_F^\Sigma} \left\{ \sum_{y \in \mathcal{N}_F^\Sigma(T)} (\tau_y, (H_y^2 - H_y^1) \times n_T) \frac{1}{3} \mathrm{Surf}(T) \right\} \end{aligned}$$

and, by interchanging the summation,

$$(\tau, (H_1 - H_2) \times n)_\Sigma \simeq \frac{1}{3} \sum_{y \in \mathcal{N}_F^\Sigma} \left(\tau_y, (H_y^2 - H_y^1) \times \left(\sum_{T \in \mathcal{T}_F^\Sigma(y)} n_T \, \mathrm{Surf}(T) \right) \right) \simeq \sum_{y \in \mathcal{N}_F^\Sigma} (\tau_y, (H_y^2 - H_y^1) \times n_y^\Sigma) \alpha_y^\Sigma, \tag{4.21}$$

where

$$\alpha_y^\Sigma = \frac{1}{3} \sum_{T \in \mathcal{T}_F^\Sigma(y)} \mathrm{Surf}(T). \tag{4.22}$$

Thus, Eq. (4.11) can be approximated by

$$\sum_{y \in \mathcal{N}_F^\Sigma} (\tau_y, (H_y^2 - H_y^1) \times n_y^\Sigma) \, \alpha_y^\Sigma = 0 \quad \forall (\tau_y)_{y \in \mathcal{N}_F^\Sigma}, \tau_y \in \mathbb{R}^3, \tau_y \times n_y^\Sigma = 0,$$

or, equivalently, by

$$(H_y^2 - H_y^1) \times n_y^\Sigma = 0 \quad \forall y \in \mathcal{N}_F^\Sigma. \tag{4.23}$$

Let us introduce the space V_0^h of the discrete fields H in V^h which satisfy the constraints (4.23):

$$V_0^h = \{H \in V^h, (H_y^2 - H_y^1) \times n_y^\Sigma = 0, \quad \forall y \in \mathcal{N}_F^\Sigma\}. \tag{4.24}$$

If the above presented approximation is made, the variational formulation (4.12), (4.9)–(4.11) can be restated:

"Find $H \in V_0^h$, $P \in L^{2h}$, $\nu \in N^{2h}$:

$$\sum_{i=1}^{2} \mu_i (H, K)_{\Omega_i} + \sum_{i=1}^{2} \Delta t \sqrt{\frac{\mu_i}{\varepsilon_i}} (H_i \times n_i, K_i \times n_i)_{\Gamma_{Ai}} + \sum_{i=1}^{2} \mu_i (\nabla \cdot K, P)_{\Omega_i} + (\nu, \mu_1 K_1 - \mu_2 K_2)_\Sigma = (G_H, K)_\Omega \quad \forall K \in V_0^h, \tag{4.25}$$

$$\sum_{i=1}^{2} \mu_i (\nabla \cdot H, q)_{\Omega_i} = 0 \quad \forall q \in L^{2h}, \tag{4.26}$$

$$(\lambda, \mu_1 H_1 - \mu_2 H_2)_\Sigma = 0 \quad \forall \lambda \in N^{2h}." \tag{4.27}$$

Now, it is well known that the constraints involved in the definition of the space V_0^h are not easily dealt with numerically. This motivates the introduction of a projection method that is detailed in the next subsection.

4.4.2. *The Projection Method*

We introduce the orthogonal projection Π (with respect to the inner product of $L^2(\Omega)$) of V^h onto V_0^h. For H in V^h, ΠH is defined by

$$\Pi H \in V_0^h, \quad (\Pi H - H, K)_\Omega = 0 \quad \forall K \in V_0^h. \tag{4.28}$$

Then, with the operators M, L, and N defined by (4.13), (4.14), and (4.15), we can write problem (4.25)–(4.27) in matrix form:

$$H \in V^h, P \in L^{2h}, \nu \in N^{2h}, \quad \Pi H = H, \quad (\Pi M \Pi) H + (\Pi L) P + (\Pi N) \nu = \Pi G_H, \quad L^{\mathrm{T}} \Pi H = 0, \quad N^{\mathrm{T}} \Pi H = 0. \tag{4.29}$$

This problem can be simplified further by introducing the generalized inverse $(\Pi M \Pi)^{-1}$ defined as the operator which to Y in V^h associates X in V^h solution of the system:

$$\Pi X = X, \quad (\Pi M \Pi) X = \Pi Y. \tag{4.30}$$

Since, the operator $\Pi M \Pi$ maps V^h into itself and has respectively V_0^h and $(V_0^h)^\perp$ for range and kernel, $(\Pi M \Pi)^{-1}$ is well defined by (4.30). Then, problem (4.29) is equivalent to

$$H = (\Pi M \Pi)^{-1} [G_H - LP - N\nu], \tag{4.31}$$

where P and ν are solutions of the system

$$(L^{\mathrm{T}} (\Pi M \Pi)^{-1} L) P + (L^{\mathrm{T}} (\Pi M \Pi)^{-1} N) \nu = L^{\mathrm{T}} (\Pi M \Pi)^{-1} G_H$$
$$(N^{\mathrm{T}} (\Pi M \Pi)^{-1} L) P + (N^{\mathrm{T}} (\Pi M \Pi)^{-1} N) \nu = N^{\mathrm{T}} (\Pi M \Pi)^{-1} G_H. \tag{4.32}$$

The size of the matrix system (4.32) has been reduced, compared with (4.19). However, this procedure is only worthwhile if the projection Π and the generalized inverse $(\Pi M \Pi)^{-1}$ can easily be computed. We now show that this is the case if we use an appropriate quadrature formula in the same spirit as for (4.21).

4.4.3. *Analytical Determination of* $(\Pi M \Pi)^{-1}$; *Mass Lumping*

The first step in the computation of Π and $(\Pi M \Pi)^{-1}$ is to replace the L^2 inner product $(H, K)_\Omega$ by a suitable approximation $(H, K)_{\Omega,h}$, for H and K belonging to V^h. According to (4.2), we can write

$$\begin{aligned}(H, K)_\Omega &= \int_\Omega H(x)K(x)\,dx \\ &= \sum_{T\in\mathscr{T}_F^\Omega} \int_T H(x)K(x)\,dx, \\ &= \sum_{T\in\mathscr{T}_F^\Omega} \sum_{y,y'\in\mathscr{N}_F^\Omega(T)} (H_y^{i(T)}, K_{y'}^{i(T)}) \int_T \phi_y^{i(T)}(x)\phi_{y'}^{i(T)}(x)\,dx,\end{aligned} \tag{4.33}$$

where $\mathscr{N}_F^\Omega(T)$ denotes the set of the nodes which are vertices of the tetrahedron T of the finer tetrahedralization and $i(T)$, the medium index of T. We recall the following quadrature formula (which is the three-dimensional version of (4.20)), exact for polynomials of degree 1,

$$\int_T g(x)\,dx \simeq \frac{1}{4}\mathrm{Vol}(T) \sum_{y \text{ vertex of } T} g(y),$$

which leads to

$$\int_T \phi_y^i(x)\phi_{y'}^i(x)\,dx \simeq \frac{1}{4}\mathrm{Vol}(T)\,\delta_{yy'}. \tag{4.34}$$

By using the previous formula in (4.33), we obtain

$$(H, K)_\Omega \simeq \sum_{T\in\mathscr{T}_F^\Omega} \sum_{y\in\mathscr{N}_F^\Omega(T)} (H_y^{i(T)}, K_y^{i(T)}) \frac{\mathrm{Vol}(T)}{4},$$

or, by exchanging the summations,

$$(H, K)_\Omega \simeq (H, K)_{\Omega,h} \tag{4.35}$$

with

$$(H, K)_{\Omega,h} = \sum_{i=1}^{2} \sum_{y\in\mathscr{N}_F^{\Omega_i}} (H_y^i, K_y^i)\,\alpha_y^{\Omega_i}, \quad \alpha_y^{\Omega_i} = \sum_{T\in\mathscr{T}_F^{\Omega_i}(y)} \frac{\mathrm{Vol}(T)}{4}, \tag{4.36}$$

where $\mathscr{T}_F^{\Omega_i}(y)$ denotes the set of tetrahedra of the finer tetrahedralization of $\overline{\Omega_i}$ which share y as a vertex. In a similar way, we can write

$$(H, K)_{\Omega_i} \simeq (H, K)_{\Omega_i,h}, \quad (H, K)_{\Omega_i,h} = \sum_{y\in\mathscr{N}_F^{\Omega_i}} (H_y^i, K_y^i)\alpha_y^{\Omega_i}. \tag{4.37}$$

Remark 4.2. In practice, the field H is identified with the vector of its values at the nodes of the tetrahedralization : $(H_y^i)_{i=1,2,y\in\mathscr{N}_F^{\Omega_i}}$ and the dot product of two such vectors is denoted by

$$\langle H, K\rangle = \sum_{i=1}^{2} \sum_{y\in\mathscr{N}_F^{\Omega_i}} (H_y^i, K_y^i).$$

Then, formula (4.36) shows that the passage from the dot product $(H, K)_{\Omega,h}$ to the dot product $\langle H, K\rangle$ is done by a diagonal matrix. In the sequel, to be consistent with the previous sections, we shall work with the former, while in practice, the latter is preferable. This diagonal property, which is also referred to as "mass lumping" is not a weak property, but it is a direct consequence of the use of a P^1-conforming method. This point was already mentioned to be crucial for the efficiency of the method in [1].

Then, in (4.28), we may replace the $L^2(\Omega)$ dot product by its approximation $(H, K)_{\Omega,h}$ and with (4.36), Π is explicitly given for any H in V^h by

$$\begin{aligned}(\Pi H)_y^i &= H_y^i \quad \forall y \notin \mathscr{N}_F^\Sigma, \\ (\Pi H)_y^i \cdot n_y^\Sigma &= H_y^i \cdot n_y^\Sigma \quad \forall y \in \mathscr{N}_F^\Sigma, i = 1, 2, \\ (\Pi H)_y^1 \times n_y^\Sigma &= (\Pi H)_y^2 \times n_y^\Sigma \\ &= \left(\sum_{i=1}^{2} \frac{\alpha_y^{\Omega_i}}{\alpha_{y^1}^{\Omega_1} + \alpha_{y^2}^{\Omega_2}} H_y^i\right) \times n_y^\Sigma \quad \forall y \in \mathscr{N}_F^\Sigma.\end{aligned} \tag{4.38}$$

Π is block diagonal, each block being associated with a node. The matrix M can also be explicitly computed. In the case where $\Gamma_{Ai} = \varnothing$, M is trivially found to be

$$(MH)_y^i = \mu_i H_y^i. \tag{4.39}$$

When $\Gamma_{Ai} \neq \varnothing$, the bilinear form $(H_i \times n_i, K_i \times n_i)_{\Gamma_{Ai}}$ can be approximated exactly in the same way as $(\tau, (H_1 - H_2) \times n)_\Sigma$ (formula (4.21)). This allows a complete analytic determination of the matrix M and of its inverse M^{-1} which is already done in [1]. In this case, the matrices M and M^{-1} are also block diagonal, each block being associated with a node. Simply, instead of being scalar, the blocks corresponding to the nodes of Γ_{Ai} are three-dimensional matrices.

It is now possible to proceed to the computation of $(\Pi M \Pi)^{-1}$. We shall only consider the case $\Gamma_{Ai} = \varnothing$ and refer to [41] for the most general case. We are concerned with the resolution of system (4.30). Since both M and Π are block diagonal, each block M_y, Π_y being associated with a node y, the matrix $(\Pi M \Pi)^{-1}$ is also block diagonal. The associated blocks will be denoted by $(\Pi M \Pi)_y^{-1}$ and are given by system (4.30) with M and Π replaced by M_y and Π_y. For an internal node $y \notin \Sigma$, M is scalar and Π is the identity of $\mathbb{R}^3$, so that the obvious result follows,

$$(\Pi M \Pi)_y^{-1} = \mu_i^{-1} \mathrm{Id}_{\mathbb{R}^3},$$

where i is the medium index of y and $\mathrm{Id}_{\mathbb{R}^3}$ is the identity matrix of $\mathbb{R}^3$. For a node y on Σ, an easy computation shows that for any given vector $Y_y = (Y_y^1, Y_y^2) \in \mathbb{R}^6$, the vector $X_y = (X_y^1, X_y^2) = (\Pi M \Pi)_y^{-1} Y_y$ is given by

$$X_y^1 = \frac{1}{|n_y^\Sigma|^2} (x_y^1 n_y^\Sigma + \xi_y \times n_y^\Sigma)$$

$$X_y^2 = \frac{1}{|n_y^\Sigma|^2} (x_y^2 n_y^\Sigma + \xi_y \times n_y^\Sigma),$$

with

$$x_y^1 = \frac{1}{\mu_1} Y_y^1 \cdot n_y^\Sigma$$

$$x_y^2 = \frac{1}{\mu_2} Y_y^2 \cdot n_y^\Sigma$$

$$\xi_y = \frac{\alpha_{y^1}^{\Omega_1} Y_y^1 + \alpha_{y^2}^{\Omega_2} Y_y^2}{\alpha_{y^1}^{\Omega_1} \mu_1 + \alpha_{y^2}^{\Omega_2} \mu_2}.$$

Therefore, the computation of the matrix $(\Pi M \Pi)^{-1}$ can be done by hand. The elimination of the Lagrange multiplier τ thus does not produce any additional computational cost and saves both computer storage (because there is no need to store τ) and computer time (because the reduced matrix system (4.32) is smaller and easier to invert than the original one (4.19)). Of course, the same procedure can be used for the E field as well, for which all the above formulas are valid, provided ε and μ are interchanged. In the next section, we investigate if a similar method can be used to eliminate the Lagrange multiplier of the transmission condition of the normal component.

4.4.4. *The Transmission Condition of the Normal Component*

If we try to mimic the method that has been used for the elimination of the Lagrange multiplier of the tangential constraints, we need to find an approximation of the bilinear form $(\nu, \mu_1 H_1 - \mu_2 H_2)_\Sigma$, for $\nu \in N^{2h}$, $H \in V^h$. We have, by splitting the integration domain Σ into triangles of the coarser tetrahedralization,

$$(\nu, \mu_1 H_1 - \mu_2 H_2)_\Sigma = \sum_{T \in \mathscr{T}_C^\Sigma} \sum_{z \in \mathscr{N}_C^\Sigma(T)} \sum_{y \in \mathscr{N}_F^\Sigma(T)} \nu_z n_z^\Sigma \times (\mu_1 H_y^1 - \mu_2 H_y^2) \int_T \psi_z^\Sigma(x) \phi_y^\Sigma(x)\, dS(x), \tag{4.40}$$

where we have denoted by $\mathscr{N}_C^\Sigma(T)$ the set of the nodes of the coarser tetrahedralization which are vertices of T and by $\mathscr{N}_F^\Sigma(T)$ the set of nodes of the finer tetrahedralization which are included in the coarse triangle T. It is fairly clear that the use of the quadrature formula (4.20) is not possible because it would lead to an unacceptable error. Therefore, we are led to consider all the integrals

$$\gamma_{zy}(T) = \int_T \psi_z^\Sigma(x) \phi_y^\Sigma(x)\, dS(x)$$

and to write formula (4.40) as

$$(\nu, \mu_1 H_1 - \mu_2 H_2)_\Sigma = \sum_{z \in \mathscr{N}_C^\Sigma} \nu_z n_z^\Sigma \cdot \left[\sum_{y \in \mathscr{N}_F^\Sigma(z)} (\mu_1 H_y^1 - \mu_2 H_y^2)\, \alpha_{zy}^\Sigma \right],$$

where $\mathscr{N}_F^\Sigma(z)$ denotes the set of nodes of the finer tetrahedralization which are included in a coarse triangle of Σ which share z as a vertex, α_{zy}^Σ is given by

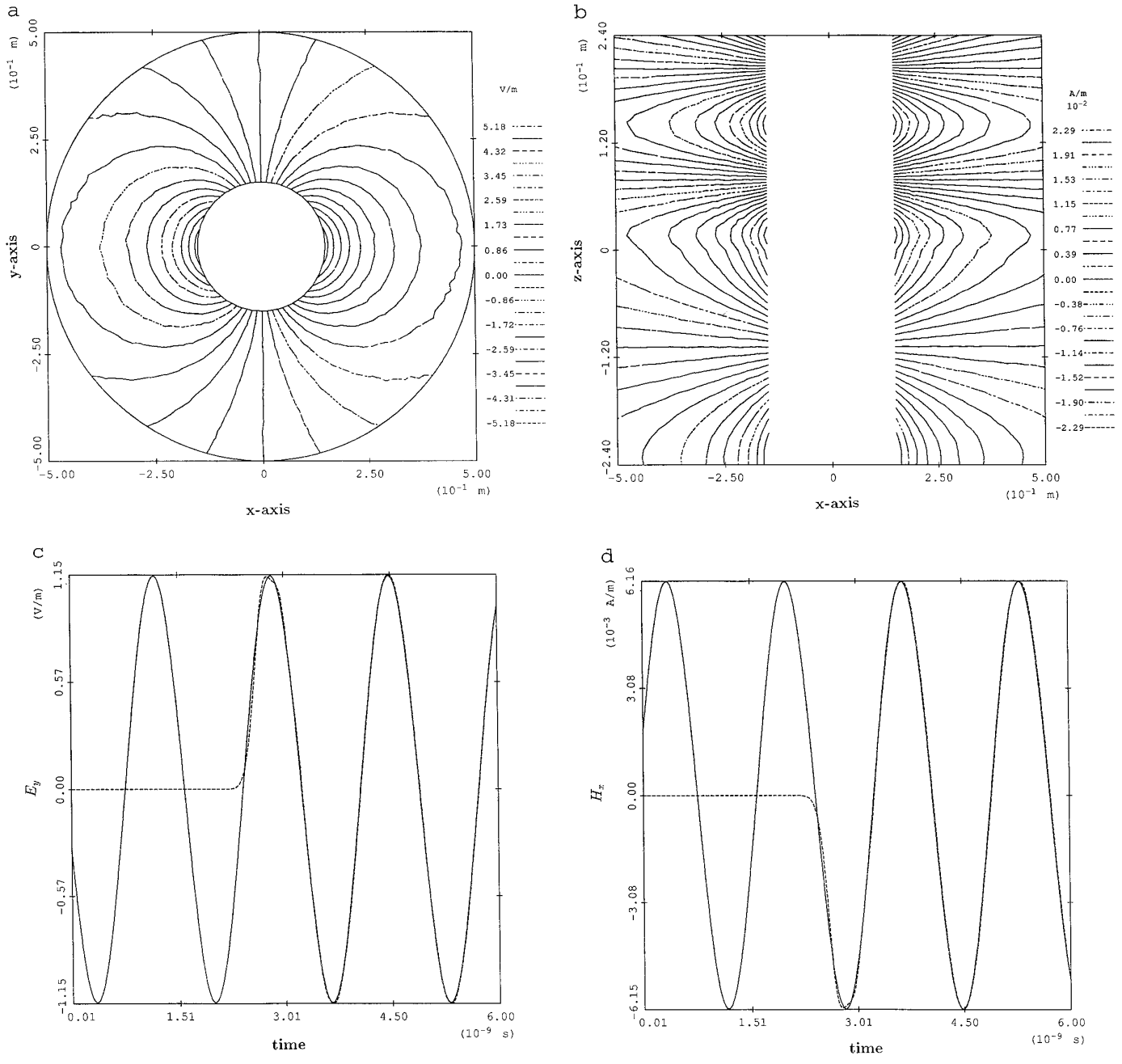

FIG. 2. (a) E_x in the (x, y)-plane, for $z = -0.192$; computed solution; (b) H_y in the (x, z)-plane, for $y = 0$; computed solution; (c) E_y in the second medium; analytic (full line) and computed (doted line) solutions; (d) H_x in the second medium; analytic (full line) and computed (dotted line) solutions.

$$\alpha_{zy}^{\Sigma} = \sum_{T \in \mathscr{T}_C^{\Sigma}(z,y)} \gamma_{zy}(T),$$

and $\mathscr{T}_C^{\Sigma}(z, y)$ stands for the set of coarse triangles on Σ which share z as a vertex and such that y is a vertex of some fine triangle included in T.

Then, the constraint

$$(\lambda, \mu_1 H_1 - \mu_2 H_2)_{\Sigma} = 0 \quad \forall \lambda \in N^{2h}$$

reduces to

$$n_z^{\Sigma} \cdot \left[\sum_{y \in \mathscr{N}_F^{\Sigma}(z)} (\mu_1 H_y^1 - \mu_2 H_y^2)\, \alpha_{xy}^{\Sigma} \right] = 0 \quad \forall z \in \mathscr{N}_C^{\Sigma}, \qquad (4.41)$$

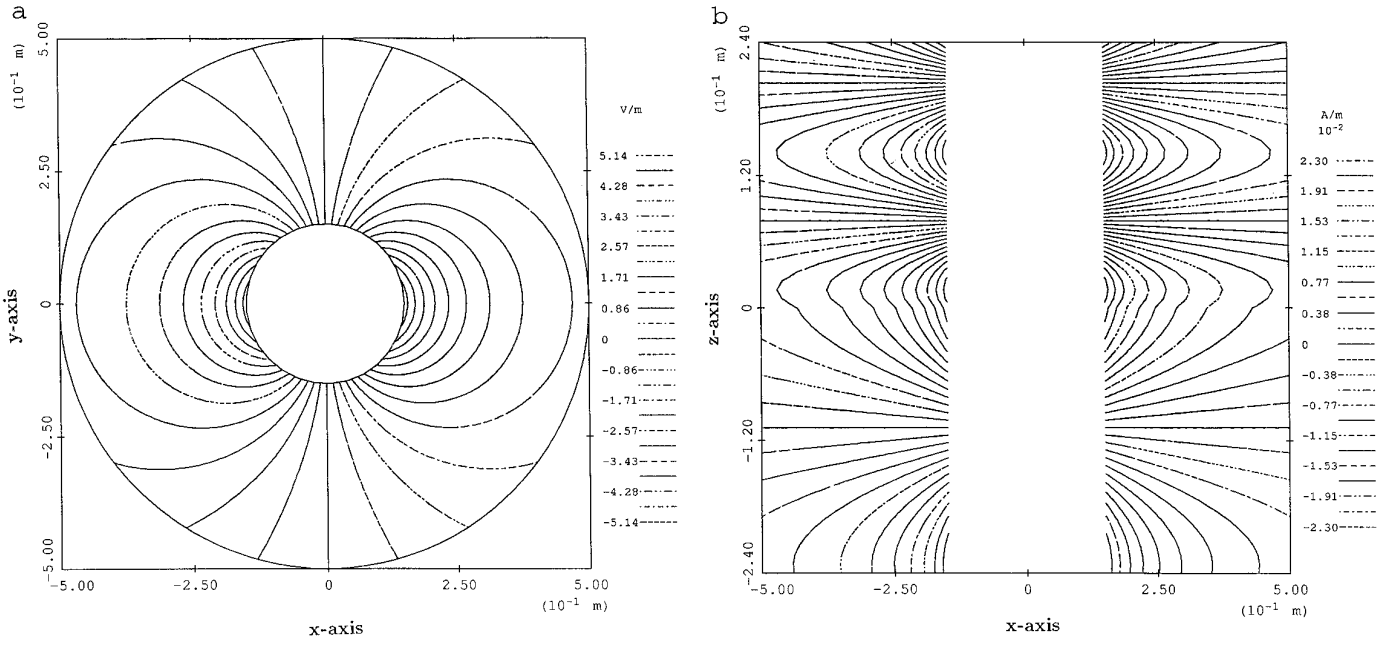

FIG. 3. (a) E_x in the (x, y)-plane, for $z = -0.192$; analytical solution; (b) H_y in the (x, z)-plane, for $y = 0$; analytical solution.

which expresses that, for any node z of the coarse triangulation of Σ, a certain mean value of the jumps of the normal components of B through Σ is zero, the mean value being computed over a certain set of fine nodes y surrounding z. However, this constraint couples all the fine nodes y together, and the projection method would not lead to an explicit inversion any longer. It is thus simpler to keep the Lagrange multiplier ν as an unknown and to directly solve system (4.32). The same conclusion is of course valid for E.

Remark 4.3. Instead of N^{2h}, we could think of approximating ν by P^1, conforming finite elements on the finer triangulation, namely,

$$N^h = \left\{ \nu \in \mathscr{C}^0(\Sigma)^3,\ \nu(x) = \sum_{y \in \mathscr{N}_F^\Sigma} \nu_y n_y^\Sigma \phi_y^\Sigma(x),\ \nu_y \in \mathbb{R},\ \forall y \in \mathscr{N}_F^\Sigma \right\}. \tag{4.42}$$

Then, the elimination of ν would have been possible in the same way as for τ and would have led to impose the transmission conditions

$$(\mu_2 H_y^2 - \mu_1 H_y^1) \cdot n_y^\Sigma = 0, \quad (\varepsilon_2 E_y^2 - \varepsilon_1 E_y^1) \cdot n_y^\Sigma = 0, \quad y \in \mathscr{N}_F^\Sigma,$$

on all the fine nodes of Σ. However, this approximation space is inconsistent with the interpretation of ν in terms of P (3.22) and with the approximation of P (4.3). The consequence is probably an overconstrained system with a slower convergence of the iterative algorithm for computing P. However, in some cases, this price can be worth being paid.

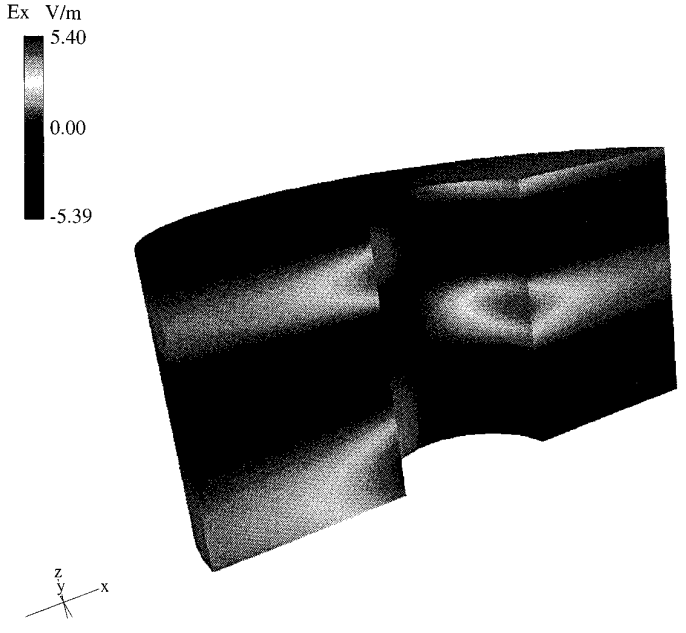

FIG. 4. Three-dimensional view of E_x; computed solution.

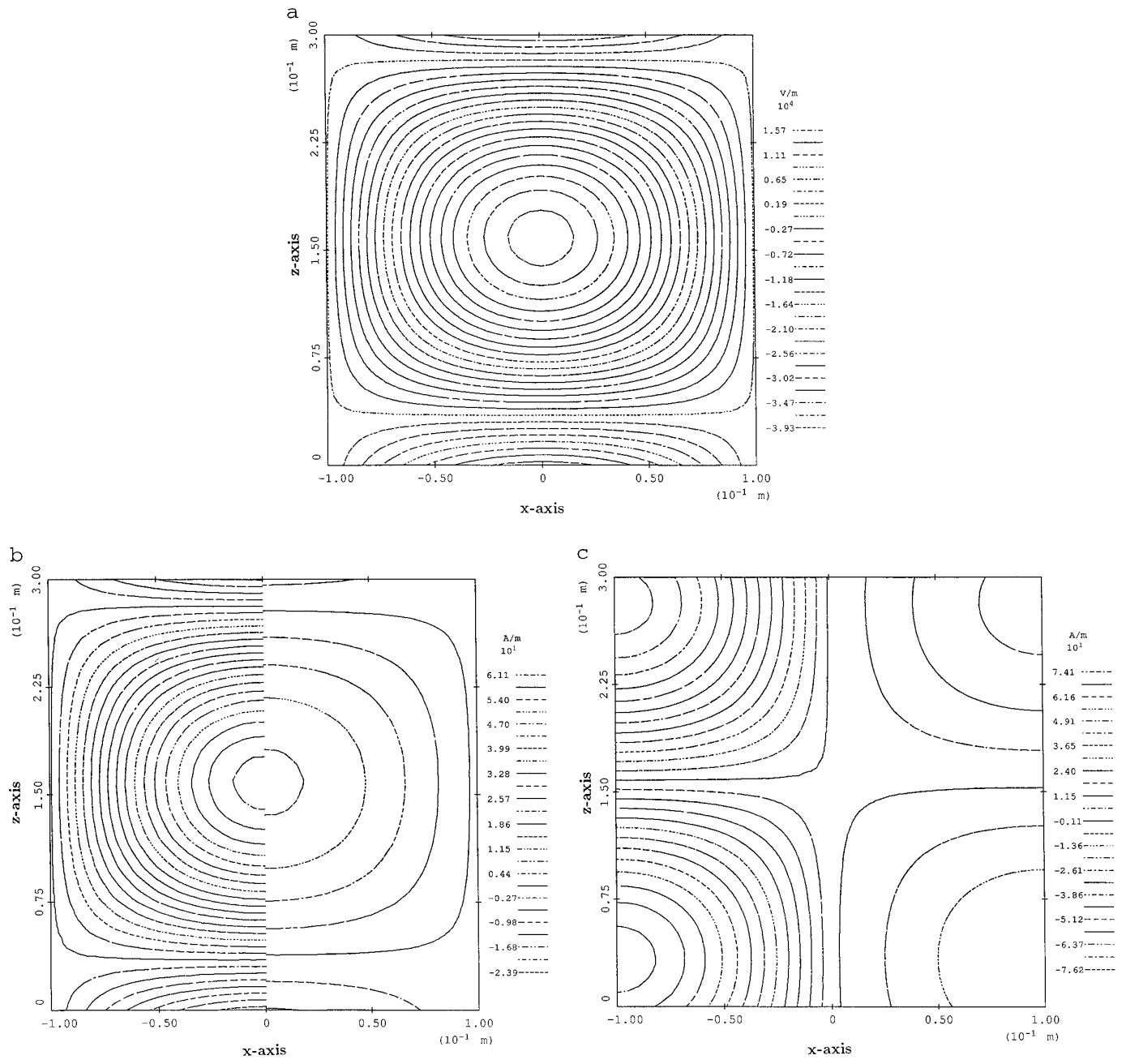

FIG. 5. (a) E_y in the (x, z)-plane, for $y = 0$; computed solution; (b) H_x in the (x, z)-plane, for $y = 0$; computed solution; (c) H_z in the (x, z)-plane, for $y = 0$; computed solution.

5. NUMERICAL RESULTS

We now give two numerical examples as a first attempt to show the validity of the proposed method. These two examples will illustrate the good behavior of the method in two very different cases (the transmission from one medium to another and the propagation through a two-media guide), both including perfect conductor and absorbing boundary conditions. A more complete study will be presented in [42] about the case of more than two media, which meet along an edge or at one point.

As a first case, we study the propagation of the TEM mode in a coaxial cylindrical waveguide, made up by the junction of two coaxes, separated by a plane interface. At time $t = 0$, we first initialize the electromagnetic fields $E(0)$ and $B(0)$ in the whole domain with zero. The coax (also discretized by irregular tetrahedra) is then illuminated by ingoing plane waves which enter normally to the lower

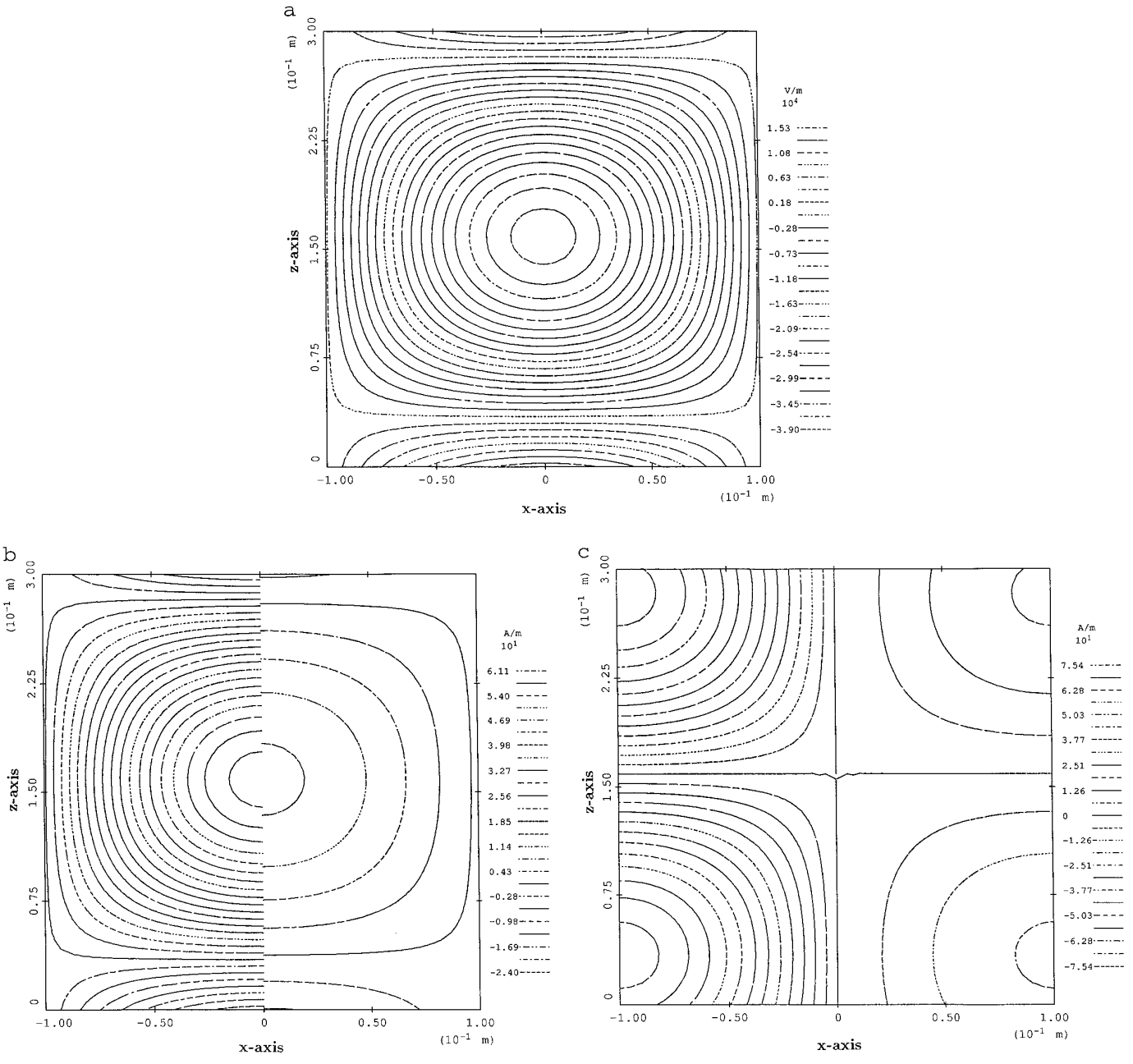

FIG. 6. (a) E_y in the (x, z)-plane, for $y = 0$; analytical solution; (b) H_x in the (x, z)-plane, for $y = 0$; analytical solution; (c) H_z in the (x, z)-plane, for $y = 0$; analytical solution.

boundary, according to the Silver–Müller boundary conditions (3.11) and (3.12). The waves first are propagated through the first medium (which is the vacuum, i.e., ε_0 and μ_0, are meeting the interface and are then propagated through the second medium (characterized by $\varepsilon/\varepsilon_0 = 4$ and $\mu/\mu_0 = 1$). This case is of interest because an analytic expression of the solution can be calculated. For this, we wrote the interface conditions at the interface between the two medium. Let us note that the wavenumbers k_i, $i = 1, 2$ and the velocities c_i, $i = 1, 2$ have to fullfil the relation $k_1/k_2 = c_2/c_1$. Figures 2a and 2b show respectively the transverse section of E_x and the longitudinal sections of H_y obtained after 300 time steps of simulation, with a Courant number of approximately 0.5. The number of points per wavelength is of about 30 in the propagation direction. These figures have to be compared with the corresponding exact solutions, depicted in Figs. 3a and 3b. Figures 2c and 2d show respectively the time variation of

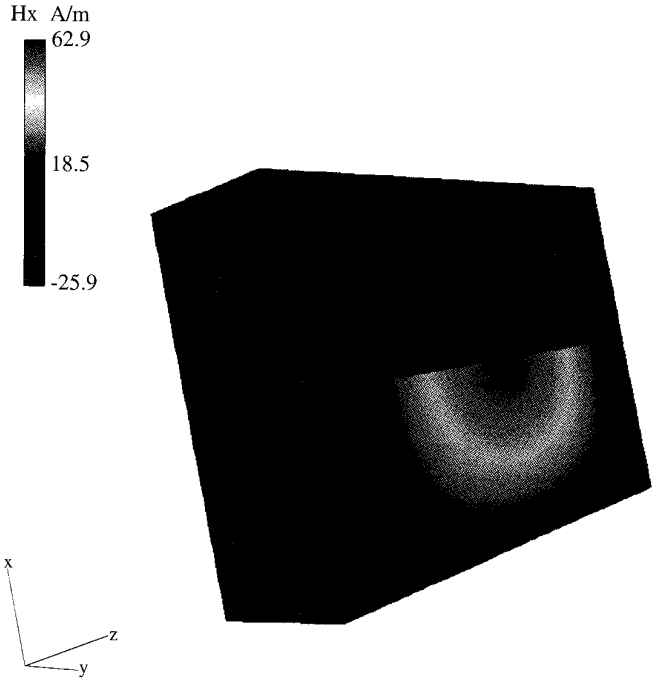

FIG. 7. Three-dimensional view of H_x; computed solution.

E_y and H_x at one point of the second medium. These solutions (dotted line) are zero (due to the initialization) until the transmitted wave reach to the considered point. On the same figure are depicted the corresponding analytic solution (full line), which do not take into account the propagation delay. One can see there is a very good agreement between the analytic and computed solutions, as soon as this point is reached by the wave. At least, Fig. 4 shows a more qualitative 3D view of the E_x computed solution.

As a second case, we study the propagation of the TE mode in a rectangular two-layer waveguide. We denote by z the direction of propagation, and the interface is included in the plane $\{x = 0\}$. The first layer ($0 < x < a$, $-b < y < b$) is an homogeneous medium with a dielectric permittivity $\varepsilon_1 = 2.210^{-12}$ and a magnetic permeability $\mu_1 = 5.610^{-6}$. Because of the velocity condition $c_1 = c_2 = c$, which is necessary to leave the TE mode to propagate in such a guide, we choose for the second layer ($-a < x < 0$, $-b < y < b$) the constants ε_2 and μ_2 such that $\varepsilon_1\mu_1 = \varepsilon_2\mu_2 = 1/c^2$. The ingoing plane waves enter normally to the face $\{z = 0\}$ in the guide, initialized with electromagnetic fields $E(0)$ and $B(0)$ equal to zero. The propagation direction is then parallel to the interface. In this case also, an analytic expression of the solution was calculated (which allowed us to obtain the velocity relation). We choose for the numerical illustration of case of y-invariant solution, which implies $H_y = E_x = 0$. Figures 5a, 5b, and 5c show respectively the components E_y, H_x, and H_z in the section $\{y = 0\}$, obtained after 900 time steps of simulation, with a Courant number and a number of points per wavelength of the same order of magnitude as in the previous case. The corresponding analytic solutions are depicted in Figs. 6a, 6b, and 6c. In the same spirit as in the first example, Fig. 7 shows a 3D view of the H_x computed solution.

6. CONCLUSION

In this paper, we have presented the treatment of interfaces between dielectric media in the framework of a P^1-conforming finite element method of approximation of the Maxwell equations. The idea is to dualize the interface conditions between the various media and to apply the standard P^1-conforming method in each subdomain where the dielectric and magnetic parameters are constant (or more generally slowly variable). If a convenient discretization of the Lagrange multipliers of these constraints is chosen, they can be eliminated (at least those corresponding to the tangential constraints) and, by using a projection technique, the interface conditions can be dealt with very little additional numerical effort. No mathematical theory has been done yet to support this method, but we think that a rigorous analysis would help to answer some of the questions that have been quoted along this paper.

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